Problem Set 12 – Relativity (`GOI36A`)

1 Angular Momentum for the Kerr Metric

1.) The only dependence that the line element ds^2 has on the coordinate ϕ is in the $d\phi^2$ term. Of course, $d\phi^2$ is invariant under constant translations along the ϕ-direction, which in turn means that the vector $\mathcal{R} = \partial_\phi$ generates an isometry of the metric. Thus, $\mathcal{R}$ is a Killing vector. Then, as discussed in class and in exercise session, we know that if we define

$$J^\mu \equiv \mathcal{R}_\nu R^{\mu\nu} \ , \tag{1.1}$$

then this vector J^μ is conserved, i.e. $\nabla_\mu J^\mu = 0$. In turn, this means we can construct an associated conserved charge. That is, the quantity

$$J \equiv -\frac{1}{8\pi G}\int_\Sigma d^3x\,\sqrt{\gamma}\,n_\mu J^\mu \ , \tag{1.2}$$

where Σ is a spacelike hypersurface with unit normal n and induced metric γ_{ab}, will be independent of which spacelike slice Σ we choose. (Note that we have put various constant factors in front; this is simply for convenience, as it will make our final answer simpler.) Intuitively, the conserved charge J associated with translations in the ϕ-direction must be the angular momentum pointing in the z-direction, since ϕ-translation invariance means that we can always rotate our system along the ϕ direction freely, i.e. angular momentum is conserved.

2.) If we now use the fact that the vector J^μ can equivalently be written as

$$J^\mu = \nabla_\nu \nabla^\mu \mathcal{R}^\nu \ , \tag{1.3}$$

then the conserved angular momentum J is hence given by

$$J = -\frac{1}{8\pi G}\int_\Sigma d^3x\,\sqrt{\gamma}\,n_\mu \nabla_\nu \nabla^\mu \mathcal{R}^\nu \ . \tag{1.4}$$

We can apply Stokes' theorem to this and turn this into an integral over the boundary $\partial\Sigma$ of our spacelike slice, due to the overall total derivative acting on $\nabla^\mu\mathcal{R}^\nu$:

$$J = -\frac{1}{8\pi G}\int_{\partial\Sigma} d^2x\,\sqrt{\gamma^{(2)}}\,n_\mu \sigma_\nu \nabla^\mu \mathcal{R}^\nu \ , \tag{1.5}$$

where σ is the unit normal to $\partial\Sigma$ and $\gamma^{(2)}_{ab}$ denotes the induced metric on $\partial\Sigma$. Thus, the angular momentum J can be written as a Komar integral.

3.) Before we proceed with the problem, we need the components of the inverse metric $g^{\mu\nu}$. Since $g_{\mu\nu}$ mixes only the t and ϕ coordinates, it is easy to see that

$$g^{rr} = \frac{1}{g_{rr}} = \frac{\Delta}{\rho^2} \ , \quad g^{\theta\theta} = \frac{1}{g_{\theta\theta}} = \frac{1}{\rho^2} \ . \tag{1.6}$$

Computing g^{tt} and $g^{\phi\phi}$ requires inverting a 2D matrix:

$$\begin{pmatrix} g^{tt} & g^{t\phi} \\ g^{\phi t} & g^{\phi\phi} \end{pmatrix} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{\phi t} & g_{\phi\phi} \end{pmatrix}^{-1} = \frac{1}{g_{tt}g_{\phi\phi} - g_{t\phi}^2}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{\phi t} & g_{tt} \end{pmatrix} \ . \tag{1.7}$$

By explicit computation, you should be able to show that

$$g_{tt}g_{\phi\phi} - g_{t\phi}^2 = -\Delta \sin^2\theta \ . \tag{1.8}$$

Thus, the inverse metric components become

$$\begin{pmatrix} g^{tt} & g^{t\phi} \\ g^{\phi t} & g^{\phi\phi} \end{pmatrix} = \frac{1}{\Delta \sin^2\theta} \begin{pmatrix} -\frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] & -\frac{2GMar\sin^2\theta}{\rho^2} \\ -\frac{2GMar\sin^2\theta}{\rho^2} & 1-\frac{2GMr}{\rho^2} \end{pmatrix} \ . \tag{1.9}$$

A good check that we have done this correctly is to note that we recover the inverse Schwarzschild metric when $a = 0$.

To compute the Komar integral for J, we first need to specify a space-like hypersurface Σ. We will choose Σ to be the hypersurface specified by $t = 0$. For any hypersurface that is specified by a condition of the form $f(x) = (\text{const.})$, we can construct an outward normal vector to this surface by computing $\xi_\mu = \nabla_\mu f$. For this $t = 0$ surface, this is given by

$$\xi_\mu = (1, 0, 0, 0) \ . \tag{1.10}$$

As expected, this is a time-like normal vector, and so Σ is in fact a space-like hypersurface. The corresponding unit normal vector n_μ is given by

$$n_\mu = -\frac{\xi_\mu}{|\xi_\mu\xi^\mu|^{1/2}} = -\frac{\xi_\mu}{|g^{tt}\xi_t\xi_t|^{1/2}} \ , \tag{1.11}$$

with the minus sign in front chosen by our normalization conventions for a spacelike hypersurface. Using our inverse metric components, we explicitly compute that

$$n_\mu = \left(-\Delta^{1/2}\rho\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]^{-1/2}, 0, 0, 0\right) \ . \tag{1.12}$$

The boundary $\partial\Sigma$ of this hypersurface is the two-sphere spanned by (θ, ϕ) as $r \to \infty$. Since a two-sphere can be obtained by fixing $r = (\text{const.})$, we can easily construct an outward-pointing normal vector to the two-sphere to be

$$\zeta_\mu = (0, 1, 0, 0) \ , \tag{1.13}$$

again using the fact that $\nabla_\mu f$ is a normal vector the surface spanned by $f = (\text{const.})$. ζ_μ is a space-like vector, and so the corresponding unit normal vector σ_μ is given by

$$\sigma_\mu = +\frac{\zeta_\mu}{|\zeta_\mu\zeta^\mu|} = +\frac{\zeta_\mu}{|g^{rr}\zeta_r\zeta_r|} \ . \tag{1.14}$$

This is easily computed:

$$\sigma_\mu = \left(0, \frac{\rho}{\sqrt{\Delta}}, 0, 0\right) \ . \tag{1.15}$$

Note that we cannot use equation (6.39) in Carroll; the presence of rotation alters the form of the normal vectors n_μ and σ_μ for Σ and $\partial\Sigma$, respectively.

Now, let $\mathcal{R} = \partial_\phi$ be the Killing vector associated with azimuthal translations. In component form,

$$\mathcal{R}^\mu = (0, 0, 0, 1) \ , \tag{1.16}$$

which means that all components are constants, and so $\partial_\mu \mathcal{R}^\nu = 0$ for all values of μ and ν. We can use this fact as well as the explicit form of our inverse metric and normal vectors to compute that

$$\begin{aligned} n_\mu \sigma_\nu \nabla^\mu \mathcal{R}^\nu &= n_\mu \sigma_\nu g^{\mu\rho} \nabla_\rho \mathcal{R}^\nu \\ &= n_\mu \sigma_\nu g^{\mu\rho} \left(\partial_\rho \mathcal{R}^\nu + \Gamma^\nu_{\rho\sigma} \mathcal{R}^\sigma \right) \\ &= n_\mu \sigma_\nu g^{\mu\rho} \Gamma^\nu_{\rho\sigma} \mathcal{R}^\sigma \\ &= n_t \sigma_r g^{t\rho} \Gamma^r_{\rho\phi} \mathcal{R}^\phi \\ &= n_t \sigma_r \left(g^{tt} \Gamma^r_{t\phi} + g^{t\phi} \Gamma^r_{\phi\phi} \right) \ . \end{aligned} \tag{1.17}$$

We don't need to evaluate both of these Christoffel symbols in general; we only need their leading-order behavior as $r \to \infty$. Similarly, we only need the leading order behavior of all other terms in the above expression. You should be able to verify that the asymptotic behaviors are

$$\begin{aligned} \Gamma^r_{t\phi} &\to -\frac{GMa \sin^2\theta}{r^2} \ , \\ \Gamma^r_{\phi\phi} &\to -r \sin^2\theta \ , \\ n_t &\to -1 \ , \\ \sigma_r &\to +1 \ , \\ g^{tt} &\to -1 \ , \\ g^{t\phi} &\to -\frac{2GMa}{r^3} \ , \end{aligned} \tag{1.18}$$

when $r \to \infty$. Putting this all together, we find that

$$n_\mu \sigma_\nu \nabla^\mu \mathcal{R}^\nu \to -\frac{3GMa \sin^2\theta}{r^2} \ , \tag{1.19}$$

as $r \to \infty$.

Now, we need to compute the determinant $\gamma^{(2)}$ of the induced metric on the two-sphere at asymptotic infinity. On a co-dimension two surface of fixed t and r, i.e. $dt = dr = 0$, the corresponding two-dimensional inducd metric is

$$ds^2_{(2)} = \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2} \left[(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta \right] d\phi^2 \ . \tag{1.20}$$

The determinant of this two-dimensional metric is simply

$$\gamma^{(2)} = \gamma^{(2)}_{\theta\theta} \gamma^{(2)}_{\phi\phi} = \sin^2\theta \left[(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta \right] \ . \tag{1.21}$$

When $r \to \infty$, the asymptotic behavior of this determinant is simply $\gamma^{(2)} \to r^4 \sin^2\theta$, and so we find that

$$\sqrt{\gamma^{(2)}} \to r^2 \sin\theta \ , \tag{1.22}$$

for the induced metric determinant on the two-sphere as $r \to \infty$.

Plugging (1.19) and (1.22) into the integral, we find that

$$\begin{aligned} J &= -\frac{1}{8\pi G} \int_{\partial\Sigma} d^2x \sqrt{\gamma^{(2)}} \, n_\mu \sigma_\nu \nabla^\mu \mathcal{R}^\nu \\ &= -\frac{1}{8\pi G} \int_0^{2\pi} d\phi \int_0^\pi d\theta \, (r^2 \sin\theta) \left(-\frac{3GMa \sin^2\theta}{r^2} \right) \\ &= \int_0^{2\pi} d\phi \int_0^\pi d\theta \left(\frac{3Ma \sin^3\theta}{8\pi} \right) \\ &= Ma \ . \end{aligned} \tag{1.23}$$

Thus, the conserved charge J associated with the Killing vector $\mathcal{R} = \partial_\phi$ is given by Ma. That is, the parameter a that appears in the Kerr metric really is the ratio J/M of angular momentum to energy. Additionally, our result is independent of the original space-like hypersurface Σ we chose to work on; J is really a *conserved* quantity.

4.) We have only considered a single component of the angular momentum because the metric is no longer spherically symmetric. The fact that the $g_{\theta\theta}$ and $g_{\phi\phi}$ components of the metric have non-trivial dependence on the parameter a and the radial coordinate r means that the Killing vectors on S^2 are no longer automatically Killing vectors for the Kerr solution. In fact, it is straightforward to show via the Killing equation $\nabla_{(\mu}K_{\nu)} = 0$ that the only S^2 Killing vector that remains a Killing vector here is ∂_ϕ. So, the rotation of the black hole has killed two of our three isometries, which means that only a single component of the angular momentum is now conserved.

2 The Ellis Wormhole

5.) When $n = 0$, this metric coincides precisely with the metric for a 4d Minkowski spacetime in spherical coordinates. The coordinate ρ must take values either on the range $(0, \infty)$ or on $(-\infty, 0)$. The $\rho = 0$ point is actually a coordinate singularity, since the size of the S^2 contracts to nothing. Thus the spacetime must end at $\rho = 0$, and so we cannot have a coordinate range for ρ that includes zero.

6.) When $|\rho| \gg 1$, the ρ^2 term dominates over n^2 in the metric, and so the spacetime once again looks like a flat Minkowski spacetime in spherical coordinates. At finite ρ, though, this is no longer true. However, the point $\rho = 0$ is no longer singular, since the size of the S^2 no longer contracts to zero. This means that we are free to take a coordinate range for ρ that includes the origin! There is thus no obstruction to letting ρ take values on the entire real line, i.e. $\rho \in (-\infty, +\infty)$. In particular, ρ can take both positive and negative values.

7.) At fixed t, the $\rho = 0$ subspace looks like a singular point when $n = 0$ and it looks like an S^2 with radius n when $n \neq 0$. So, this subspace is zero-dimensional for $n = 0$, but two-dimensional for $n \neq 0$. This is what allows us to extend the range of ρ to the entire real line when $n \neq 0$.

8.) The picture should be something like what is shown in figure 1. At $\rho = \pm\infty$, the spacetime asymptotes to the usual flat spacetime in spherical coordinates. At the center, i.e. $\rho = 0$, the spacetime doesn't hit a singularity; instead, the size of the S^2 in the metric is finite, and so we can keep going through, thus connecting the two asymptotic flat regions together.

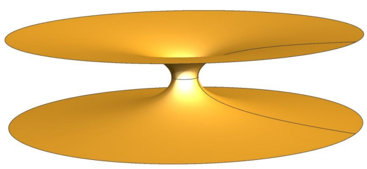

Figure 1. A schematic depiction of the spacetime when $n \neq 0$. The vertical axis corresponds to the ρ direction. This is the "Ellis wormhole".

9.) So far our "wormhole" is effectively a bridge between two different Minkowski spacetimes, one that is located at $\rho = \infty$ and one that is located at $\rho = -\infty$. If we wanted the wormhole to be a shortcut between two points in the same spacetime, we would need some way to connect the two asymptotic Minkowski spacetimes together, as depicted in figure 2. In fact, if we do this extension such that the spacetime remains flat everywhere except for near the actual bridge, then the bridge really does act as a shortcut between points!

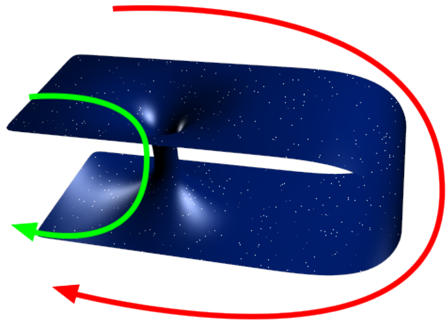

Figure 2. A schematic depiction of the spacetime when we have additionally glued together the two asymptotic Minkowski regions. We can go the long way around to go from one to the other, or we can traverse the wormhole for a shortcut.

10.) The metric is given by

$$ds^2 = -dt^2 + d\rho^2 + \left(\rho^2 + n^2\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \ . \tag{2.1}$$

The corresponding inverse metric components of the metric are

$$g^{tt} = -1 \ , \quad g^{\rho\rho} = 1 \ , \quad g^{\theta\theta} = \frac{1}{\rho^2 + n^2} \ , \quad g^{\phi\phi} = \frac{1}{(\rho^2 + n^2)\sin^2\theta} \ , \tag{2.2}$$

with all other components vanishing. Since $g_{tt} = g^{tt} = -1$, which is a constant, and no components of the metric depend on t, we can immediately see that any Christoffel symbol that has t for at least one of its components must vanish. Thus, to compute the Christoffel symbols, we only need to consider letting the components take values among ρ, θ, and ϕ.

Notice the metric and the inverse metric are both diagonal. Therefore the only way to get a non-trivial Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ is to have one of the indices appear at least twice. However, we cannot have any index repeated three times; $g_{\rho\rho}$ is indepenent of ρ, $g_{\theta\theta}$ is independent of θ, and $g_{\phi\phi}$ is independent of ϕ, and so $\partial_\mu g_{\nu\rho} = 0$ when all indices are equal. So, all of the non-trivial Christoffel symbols will have one index repeated exactly twice. Additionally, all components of the metric depend on ρ or θ, not ϕ. So, symbols like $\Gamma^{\phi}_{\rho\rho} = \frac{1}{2}g^{\phi\phi}\left(-\partial_\phi g_{\rho\rho}\right)$ and $\Gamma^{\phi}_{\theta\theta} = \frac{1}{2}g^{\phi\phi}\left(-\partial_\phi g_{\theta\theta}\right)$ will vanish automatically. Similarly, so will $\Gamma^{\rho}_{\phi\rho}$ and $\Gamma^{\theta}_{\phi\theta}$. And, since $g_{\rho\rho}$ is constant and doesn't depend on θ, $\Gamma^{\rho}_{\theta\rho}$ and $\Gamma^{\theta}_{\rho\rho}$ will also vanish by similar reasoning. Lastly, the Christoffel symbols are symmetric in their lower indices. Taking this all into account, the non-trivial independent Christoffel symbols are the following:

$$\begin{aligned} &\Gamma^{\rho}_{\theta\theta} \ , \quad \Gamma^{\rho}_{\phi\phi} \ , \\ &\Gamma^{\theta}_{\rho\theta} \ , \quad \Gamma^{\theta}_{\phi\phi} \ , \\ &\Gamma^{\phi}_{\rho\phi} \ , \quad \Gamma^{\phi}_{\theta\phi} \ . \end{aligned} \tag{2.3}$$

All others vanish, or can be obtained by symmetry (e.g. $\Gamma^{\theta}_{\theta\rho} = \Gamma^{\theta}_{\rho\theta}$). Explicitly, their computation is as follows:

$$
\begin{aligned}
\Gamma^{\rho}_{\theta\theta} &= \frac{1}{2} g^{\rho\rho} \left(-\partial_{\rho} g_{\theta\theta}\right) = -\rho \ , \\
\Gamma^{\rho}_{\phi\phi} &= \frac{1}{2} g^{\rho\rho} \left(-\partial_{\rho} g_{\phi\phi}\right) = -\rho \sin^2\theta \ , \\
\Gamma^{\theta}_{\rho\theta} &= \frac{1}{2} g^{\theta\theta} \left(\partial_{\rho} g_{\theta\theta}\right) = \frac{\rho}{\rho^2+n^2} \ , \\
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2} g^{\theta\theta} \left(-\partial_{\theta} g_{\phi\phi}\right) = -\sin\theta\cos\theta \ , \\
\Gamma^{\phi}_{\rho\phi} &= \frac{1}{2} g^{\phi\phi} \left(\partial_{\rho} g_{\phi\phi}\right) = \frac{\rho}{\rho^2+n^2} \ , \\
\Gamma^{\phi}_{\theta\phi} &= \frac{1}{2} g^{\phi\phi} \left(\partial_{\theta} g_{\phi\phi}\right) = \cot\theta \ .
\end{aligned}
\tag{2.4}
$$

Note that we heavily used the fact that the metric is diagonal when writing down these expressions. In particular, it is very important that $g^{\rho\mu}\partial_{\mu} = g^{\rho\rho}\partial_{\rho}$, $g^{\theta\mu}\partial_{\mu} = g^{\theta\theta}\partial_{\theta}$, etc.

Now, we note that none of the Christoffel symbols depend on the coordinate t, and any Christoffel symbol with t as one of its components must vanish. Therefore any component of the Riemann tensor ${R^{\mu}}_{\nu\lambda\sigma}$ with t appearing as one of its indices must vanish. And, since the metric is diagonal and also independent of t, the same must also be true for $R_{\mu\nu\lambda\sigma}$. Thus, to obtain the non-trivial Riemann tensor components, we still only need to let indices take values among ρ, θ, and ϕ. From this, and using the (anti-)symmetry properties of the Riemann tensor, we are left with the following non-trivial symbols:

$$
\begin{aligned}
&R_{\rho\theta\rho\theta} \ , \quad R_{\rho\phi\rho\phi} \ , \quad R_{\theta\phi\theta\phi} \ , \\
&R_{\rho\theta\rho\phi} \ , \quad R_{\theta\rho\theta\phi} \ , \quad R_{\phi\rho\phi\theta} \ .
\end{aligned}
\tag{2.5}
$$

All of these are such that one index must appear exactly twice and the others once, none of them vanish by antisymmetry, and all other non-zero Riemann tensor components are obtained by swapping around indices from this set of six.

We can compute these explicitly as follows. First, we compute:

$$
\begin{aligned}
R_{\rho\theta\rho\theta} &= g_{\rho\rho} {R^{\rho}}_{\theta\rho\theta} \\
&= (1) \left(\partial_{\rho}\Gamma^{\rho}_{\theta\theta} - \partial_{\theta}\Gamma^{\rho}_{\theta\rho} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\theta\theta} - \Gamma^{\rho}_{\theta\lambda}\Gamma^{\lambda}_{\theta\rho}\right) \\
&= \partial_{\rho}\left(-\rho\right) - \partial_{\theta}\left(0\right) + 0 - \Gamma^{\rho}_{\theta\theta}\Gamma^{\theta}_{\rho\theta} \\
&= -1 + \frac{\rho^2}{\rho^2+n^2} \\
&= -\frac{n^2}{\rho^2+n^2} \ .
\end{aligned}
\tag{2.6}
$$

Note that the index λ is summed over. All of the values it takes yield zero in the third term, while only the $\lambda = \theta$ component in the fourth term yields non-zero Christoffel symbols. The next is similar:

$$
\begin{aligned}
R_{\rho\phi\rho\phi} &= g_{\rho\rho} {R^{\rho}}_{\phi\rho\phi} \\
&= (1) \left(\partial_{\rho}\Gamma^{\rho}_{\phi\phi} - \partial_{\phi}\Gamma^{\rho}_{\phi\rho} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\rho}_{\phi\lambda}\Gamma^{\lambda}_{\phi\rho}\right) \\
&= \partial_{\rho}\left(-\rho\sin^2\theta\right) - \partial_{\phi}\left(0\right) + 0 - \Gamma^{\rho}_{\phi\phi}\Gamma^{\phi}_{\phi\rho} \\
&= -\sin^2\theta + \frac{\rho\sin^2\theta}{\rho^2+n^2} \\
&= -\frac{n^2\sin^2\theta}{\rho^2+n^2} \ .
\end{aligned}
\tag{2.7}
$$

The third Riemann tensor component is probably the most non-trivial computation:

$$
\begin{aligned}
R_{\theta\phi\theta\phi} &= g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} \\
&= \left(\rho^2 + n^2\right) \left(\partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\phi\theta} + \Gamma^\theta_{\theta\lambda} \Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda} \Gamma^\lambda_{\phi\theta} \right) \\
&= \left(\rho^2 + n^2\right) \left(\partial_\theta \left(-\sin\theta \, \cos\theta\right) - \partial_\phi \left(0\right) + \Gamma^\theta_{\theta\rho} \Gamma^\rho_{\phi\phi} - \Gamma^\theta_{\phi\phi} \Gamma^\phi_{\phi\theta} \right) \\
&= \left(\rho^2 + n^2\right) \left(-\cos^2\theta + \sin^2\theta - \frac{\rho^2 \sin^2\theta}{\rho^2 + n^2} + \cos^2\theta \right) \\
&= n^2 \sin^2\theta \ .
\end{aligned} \tag{2.8}
$$

The remaining three independent components of the Riemann tensor all vanish. The computation follows similarly to the ones done above, albeit slightly easier since most terms are zero, so I will leave this out for now. We therefore have the following non-zero independent components of the Riemann tensor:

$$
R_{\rho\theta\rho\theta} = -\frac{n^2}{\rho^2 + n^2} \ , \quad R_{\rho\phi\rho\phi} = -\frac{n^2 \sin^2\theta}{\rho^2 + n^2} \ , \quad R_{\theta\phi\theta\phi} = n^2 \sin^2\theta \ . \tag{2.9}
$$

Since any component of the Riemann tensor where more than two different indices appear vanishes, it is clear that all off-diagonal components of the Ricci tensor must be zero. The diagonal components are easily computed:

$$
\begin{aligned}
R_{tt} &= g^{\mu\nu} R_{\mu t \nu t} = 0 \ , \\
R_{\rho\rho} &= g^{\mu\nu} R_{\mu\rho\nu\rho} = g^{\theta\theta} R_{\theta\rho\theta\rho} + g^{\phi\phi} R_{\phi\rho\phi\rho} = -\frac{2n^2}{(\rho^2 + n^2)^2} \ , \\
R_{\theta\theta} &= g^{\mu\nu} R_{\mu\theta\nu\theta} = g^{\rho\rho} R_{\rho\theta\rho\theta} + g^{\phi\phi} R_{\phi\theta\phi\theta} = 0 \ , \\
R_{\phi\phi} &= g^{\mu\nu} R_{\mu\phi\nu\phi} = g^{\rho\rho} R_{\rho\phi\rho\phi} + g^{\theta\theta} R_{\theta\phi\theta\phi} = 0 \ .
\end{aligned} \tag{2.10}
$$

That is, only $R_{\rho\rho}$ ends up being non-zero. The Ricci scalar is then simply given by

$$
R = g^{\rho\rho} R_{\rho\rho} = -\frac{2n^2}{(\rho^2 + n^2)^2} \ . \tag{2.11}
$$

Finally, putting this all together, we can compute G_{tt}:

$$
G_{tt} = R_{tt} - \frac{1}{2} g_{tt} R = -\frac{1}{2} \left(-1\right) \left(-\frac{2n^2}{(\rho^2 + n^2)^2} \right) = -\frac{n^2}{(\rho^2 + n^2)^2} \ . \tag{2.12}
$$

This is exactly what we set out to prove. Since $G_{tt} < 0$, the Einstein equation then tells us that $T_{tt} < 0$ as well. This is problematic, because the (t, t) component of the stress-energy tensor corresponds to the energy density of the matter in our spacetime. In order for it to be negative, we would somehow need some kind of "negative" density of energy. It is clearly unphysical to talk about a "negative" energy flux, and so we cannot realistically satisfy the Einstein equation for the Ellis wormhole solution.